# Featured Weighted Automata

Uli Fahrenberg Axel Legay

École polytechnique, Palaiseau, France Inria Rennes, France

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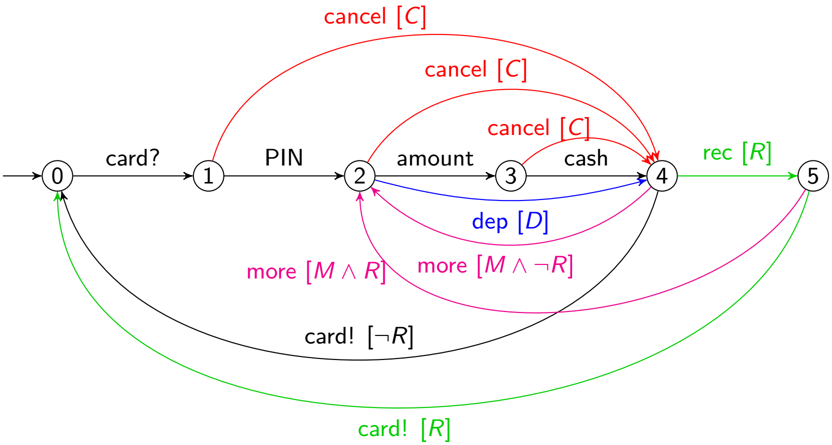
# Elevator Pitch

- featured transition systems for modeling software product lines
  - transitions can be turned on and off depending on available features
- weighted automata for modeling quantitative systems
  - shortest path; maximum flow; energy consumption; probabilities
- here: featured weighted automata for quantitative properties of SPLs
- key lemma: A featured weighted automaton is a weighted automaton

# Motivation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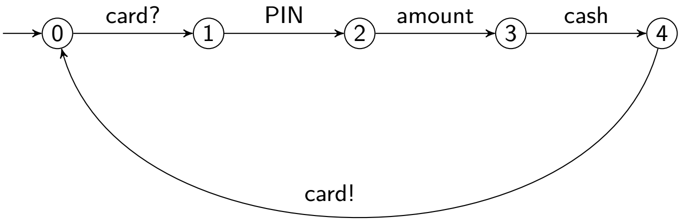

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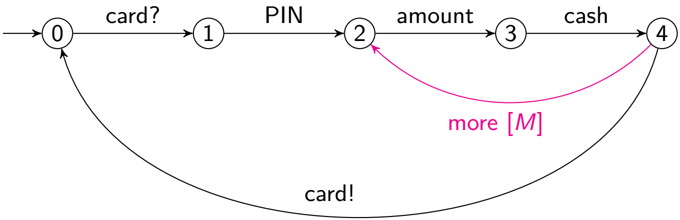

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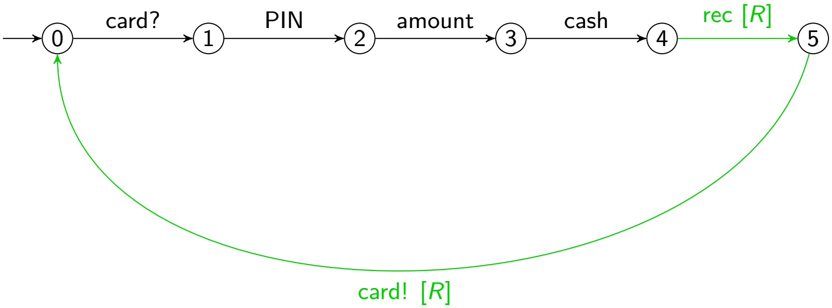

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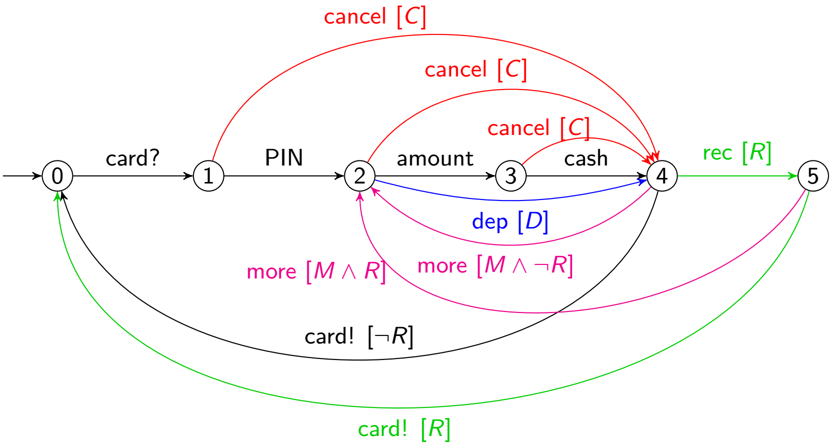
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# Motivation

Quantitative analysis of FTS:

- Cordy, Schobbens, Heymans, Legay ICSE 2013: *Beyond Boolean product-line model checking*
- Olaechea, U.F., Atlee, Legay SPLC 2016: *Long-term average cost in featured transition systems*

Here: Generalization to semiring-weighted FTS

- semiring-weighted automata for modeling and analysis of different types of quantitative properties
- Kleene algebra
- key lemma: An FTS on products $px$ weighted in a semiring $K$ is an automaton weighted in the function semiring $px \to K$

## Semirings

A semiring is a structure $(K, \oplus, \otimes, 0, 1)$ such that

- $(K, \oplus, 0)$ is a commutative monoid,
  - $x \oplus y = y \oplus x$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, $x \oplus 0 = x$
- $(K, \otimes, 1)$ is a monoid,
  - $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes 1 = 1 \otimes x = x$
- and which satisfies distributive and annihilation laws:
  - $x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$, $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
  - $x \otimes 0 = 0 \otimes x = 0$

Examples:

- natural numbers: $(\mathbb{N}, +, \cdot, 0, 1)$
- languages over some alphabet $\Sigma$: $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
- Boolean semiring: $(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$

# Semiring-Weighted Automata

Let $K$ be a semiring. A $K$-weighted automaton is a tuple $\mathcal{S} = (S, I, F, T)$:

- $S$ finite set of states, $I \subseteq S$ initial, $F \subseteq S$ accepting
- $T \subseteq S \times K \times S$ finite set of transitions

An accepting path in $\mathcal{S}$: finite sequence $\pi = (s_0, x_0, s_1, \dots, x_k, s_{k+1})$ of transitions $(s_0, x_0, s_1), \dots, (s_k, x_k, s_{k+1}) \in T$, with $s_0 \in I$ and $s_{k+1} \in F$

- weight of $\pi$: $w(\pi) = x_0 \otimes \cdots \otimes x_k$

The reachability value of $\mathcal{S}$:

$$|\mathcal{S}| = \bigoplus \{ w(\pi) \mid \pi \text{ accepting path in } \mathcal{S} \}$$

- if this infinite sum exists in $K$

# Semiring-Weighted Automata: Examples

Recall

- for $\pi = (s_0, x_0, s_1, \dots, x_k, s_{k+1})$: $w(\pi) = x_0 \otimes \cdots \otimes x_k$
- $|\mathcal{S}| = \bigoplus\{w(\pi) \mid \pi \text{ accepting path in } \mathcal{S}\}$

Boolean semiring $(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$:

- $w(\pi) = \mathbf{tt}$ iff all $x_i = \mathbf{tt}$
- $|\mathcal{S}| = \mathbf{tt}$ iff an accepting state is reachable (through $\mathbf{tt}$-labeled transitions)

Tropical semiring $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$:

- $w(\pi) = x_0 + \cdots + x_k$
- $|\mathcal{S}| =$ minimum reachability value / shortest path

Fuzzy semiring $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \max, \min, 0, \infty)$:

- $w(\pi) = \min\{x_0, \dots, x_k\}$
- $|\mathcal{S}| =$ maximum flow

# Conway Semirings

A star semiring is a semiring $(K, \oplus, \otimes, 0, 1)$ with a star operation $^* : K \to K$

- intuition: $\oplus$ for choice, $\otimes$ for composition, $^*$ for iteration

A Conway semiring is a star semiring $(K, \oplus, \otimes, ^*, 0, 1)$ in which

$$(x \otimes y)^* = 1 \oplus x \otimes (y \otimes x)^* \otimes y$$
$$(x \oplus y)^* = (x^* \otimes y)^* \otimes x^*$$

- encodes properties of iteration

Examples:

- Boolean: $x^* = \mathbf{tt}$
- Tropical: $x^* = 0$
- Fuzzy: $x^* = \infty$

# Matrix Semirings

Let $K$ be a semiring and $n \geq 1$. The matrix semiring over $K$ is $(K^{n\times n}, \oplus, \otimes, \mathbf{0}, \mathbf{1})$

- standard matrix addition and multiplication, like in linear algebra; $\mathbf{0}$ = zero matrix, $\mathbf{1}$ = identity matrix

Old theorem: If $K$ is a Conway semiring, then so is $K^{n\times n}$

- with $M^*_{i,j} = \bigoplus_{m\geq 0} \bigoplus_{1\leq k_1,\dots,k_m\leq n} M_{i,k_1} \otimes M_{k_1,k_2} \otimes \cdots \otimes M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^* = \begin{bmatrix} (a \oplus b \otimes d^* \otimes c)^* & (a \oplus b \otimes d^* \otimes c)^* \otimes b \otimes d^* \\ (d \oplus c \otimes a^* \otimes b)^* \otimes c \otimes a^* & (d \oplus c \otimes a^* \otimes b)^* \end{bmatrix}$$

(recursively)

# Matrix Semirings

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generalized Floyd-Warshall

(recursively)

# Automata Weighted in Conway Semirings

Let $K$ be a Conway semiring

- a $K$-weighted automaton (with $n$ states): $\mathcal{S} = (\alpha, M, \kappa)$
- $\alpha \in \{0,1\}^n$ initial vector, $\kappa \in \{0,1\}^n$ accepting vector, $M \in S^{n \times n}$ transition matrix
- (equivalent to representation $\mathcal{S} = (S, I, F, T)$ )
- Recall $|\mathcal{S}| = \bigoplus \{ w(\pi) \mid \pi \text{ accepting path in } \mathcal{S} \}$ (if it exists in $K$)
- Old theorem: $|\mathcal{S}|$ exists in $K$, and $|\mathcal{S}| = \alpha M^* \kappa$

# Featured Weighted Automata

Let $K$ be a Conway semiring and $px$ a set of products. A featured $K$-weighted automaton is a tuple $\mathcal{F} = (S, I, F, T)$:

- $S$ finite set of states, $I \subseteq S$ initial, $F \subseteq S$ accepting
- $T \subseteq S \times [px \to K] \times S$ finite set of transitions

Let $p \in px$. The projection of $\mathcal{F}$ to $p$ is the $K$-weighted automaton $\text{proj}_p(\mathcal{F}) = (S, I, F, T')$, with $T' = \{(s, f(p), s') \mid (s, f, s') \in T\}$

- We are interested in the values $|\text{proj}_p(\mathcal{F})|$ for all $p \in px$

Theorem: $[px \to K]$ is a Conway semiring, with $(f \oplus g)(p) = f(p) \oplus g(p)$, $(f \otimes g)(p) = f(p) \otimes g(p)$, and $f^*(p) = (f(p))^*$, and for all $p \in px$,

$$|\text{proj}_p(\mathcal{F})| = |\mathcal{F}|(p)$$

- family-based analysis: compute $|\mathcal{F}|$

# Featured Weighted Automata, Symbolically

Recall: A featured $K$-weighted automaton is a tuple $\mathcal{F} = (S, I, F, T)$ with $T \subseteq S \times [px \to K] \times S$

- for computations in semiring $[px \to K]$, need good symbolic representation of functions $px \to K$

Let $N$ be a set of features, so that $px \subseteq 2^N$.

- $\mathbb{B}(N)$: Boolean expressions over $N$ (feature guards)
- for $\gamma \in \mathbb{B}(N)$: $[\![\gamma]\!]$ = all products which satisfy $\gamma$
- guard partition: $P \subseteq \mathbb{B}(N)$ such that $[\![\bigvee P]\!] = px$, $\forall \gamma \in P : [\![\gamma]\!] \neq \emptyset$, and $\forall \gamma_1 \neq \gamma_2 \in P : [\![\gamma_1]\!] \cap [\![\gamma_2]\!] = \emptyset$

Let $GP[K] = \{f : P \to K \mid P \text{ guard partition}, \forall \gamma_1 \neq \gamma_2 \in P : f(\gamma_1) \neq f(\gamma_2)\}$

- injective functions from guard partitions to $K$

# Featured Weighted Automata, Computationally

$$(f \oplus g)(p) = f(p) \oplus g(p)$$

**function** $K\textsc{Sum}(f_1 : P_1 \to K, f_2 : P_2 \to K)$: $GP[K]$
    **var** $f'$, $P'$
    $P' \leftarrow \emptyset$
    **for all** $\gamma_1 \in P_1$ **do**
        **for all** $\gamma_2 \in P_2$ **do**
            **if** $[\![\gamma_1 \wedge \gamma_2]\!] \neq \emptyset$ **then**
                $P' \leftarrow P' \cup \{\gamma_1 \wedge \gamma_2\}$
                $f'(\gamma_1 \wedge \gamma_2) \leftarrow f_1(\gamma_1) \oplus f_2(\gamma_2)$
    **return** $K\textsc{Combine}(f')$

# Featured Weighted Automata, Computationally

$$(f \otimes g)(p) = f(p) \otimes g(p)$$

**function** $K\textsc{Prod}(f_1 : P_1 \to K, f_2 : P_2 \to K)$: $GP[K]$
    **var** $f'$, $P'$
    $P' \leftarrow \emptyset$
    **for all** $\gamma_1 \in P_1$ **do**
        **for all** $\gamma_2 \in P_2$ **do**
            **if** $[\![\gamma_1 \wedge \gamma_2]\!] \neq \emptyset$ **then**
                $P' \leftarrow P' \cup \{\gamma_1 \wedge \gamma_2\}$
                $f'(\gamma_1 \wedge \gamma_2) \leftarrow f_1(\gamma_1) \otimes f_2(\gamma_2)$
    **return** $K\textsc{Combine}(f')$

# Featured Weighted Automata, Computationally

$$f^*(p) = (f(p))^*$$

**function** $K\textsc{Star}(f : P \to K)$: $GP[K]$
    **var** $f'$
    **for all** $\gamma \in P$ **do**
        $f'(\gamma) \leftarrow f(\gamma)^*$
    **return** $K\textsc{Combine}(f')$

# Featured Weighted Automata, Computationally

**function** $K\textsc{Combine}(f : P \to K)$: $GP[K]$

    **var** $\tilde{f}$, $\tilde{P}$

    $\tilde{P} \leftarrow \emptyset$

    **while** $P \neq \emptyset$ **do**

        Pick and remove $\gamma$ from $P$

        $x \leftarrow f(\gamma)$

        **for all** $\delta \in P$ **do**

            **if** $f(\delta) = x$ **then**

                $\gamma \leftarrow \gamma \vee \delta$

                $P \leftarrow P \setminus \{\delta\}$

        $\tilde{P} \leftarrow \tilde{P} \cup \{\gamma\}$

        $\tilde{f}(\gamma) \leftarrow x$

    **return** $\tilde{f} : \tilde{P} \to K$

# Conclusion

- family-based analysis of featured weighted automata is easy in theory
  - because featured weighted automata are weighted automata
- in practice:
  - Cordy, Schobbens, Heymans, Legay ICSE 2013: featured shortest paths
  - Olaechea, U.F., Atlee, Legay SPLC 2016: featured long-term average
- both show that family-based analysis is better than product-based
  - but not always, and not much
  - problem: partition splitting

International Workshop on

# Methods and Tools for Distributed Hybrid Systems

Aalborg, Denmark, 25-26 August 2017, associated with MFCS 2017

The purpose of DHS is to connect people working in *real-time* systems, *hybrid* systems, *control* theory, *distributed* computing, and *concurrency*, in order to advance the subject of **distributed hybrid systems**.

Distributed hybrid systems, or distributed *cyber-physical* systems, are abundant, but ensuring their correct functioning is very difficult. We believe that convergence and interaction of methods and tools from different areas of *computer science*, *engineering*, and *mathematics* is needed in order to advance the subject.

This first edition of the DHS workshop aims at gathering researchers which work in the above areas in order to facilitate collaboration and discuss how the subject may advance.

## Invited Speakers

Martin Fränzle
Oldenburg, Germany

Kim G. Larsen
Aalborg, Denmark

Sergio Rajsbaum
Mexico City, Mexico

Martin Raussen
Aalborg, Denmark

Rafael Wisniewski
Aalborg, Denmark

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